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Note

A Generalization of an Inequality of Stepanov

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Let $\Gamma_n(p)$ denote a random graph with n vertices in which any two vertices, independently of the others, are connected by an edge with probability $p = 1 - q$ where $0 \leqslant q \leqslant 1$. Denote by $Q_n(q)$ the probability that $\Gamma_n(p)$ is connected. In 1970 V. E. Stepanov proved that $Q_n(q) \leqslant (1 - q^n)^{n-1}$ for $n \geqslant 1$. In this paper a simple proof is given for a general inequality related to the connectedness of $\Gamma_n(p)$ which contains Stepanov's inequality as a particular case.

1. INTRODUCTION

Let us consider a simple graph with vertex set $\{1, 2, ..., n\}$. Define $\delta_{ij} = 1$ if vertices i and j are joined by an edge, and $\delta_{ij} = 0$ if vertices i and j are not joined by an edge. Let us assume that δ_{ij}, $1 \leqslant i < j \leqslant n$, are mutually independent random variables for which

$$\mathbf{P}\{\delta_{ij} = 1\} = p \qquad \text{and} \qquad \mathbf{P}\{\delta_{ij} = 0\} = q, \tag{1}$$

where $p \geqslant 0$, $q \geqslant 0$, and $p + q = 1$. We shall use the abbreviation $\Gamma_n(p)$ for this model.

Denote by $Q_n(q)$ the probability that the random graph $\Gamma_n(p)$ is connected. A graph is said to be connected if any two different vertices are joined by a path (an uninterrupted sequence of edges). We have $Q_1(q) = 1$, $Q_2(q) = 1 - q$, $Q_3(q) = 1 - 3q^2 + 2q^3$, $Q_4(q) = 1 - 4q^3 - 3q^4 + 12q^5 - 6q^6$, ...

In 1959 E. N. Gilbert [1] demonstrated that

$$\sum_{i=1}^{n} \binom{n-1}{i-1} q^{i(n-i)} Q_i(q) = 1 \tag{2}$$

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for $n=1, 2, \ldots$. From (2) it is easy to deduce that

$$1-Q_n(q) \leqslant \sum_{i=1}^{n-1} \binom{n-1}{i-1} q^{i(n-i)} \leqslant q^{n/2}[(1+q^{(n-2)/2})^n - 1] \qquad (3)$$

if $n \geqslant 2$. By (3) we have a good lower bound for $Q_n(q)$ if $q \to 0$. In 1970 V. E. Stepanov [6] found a good upper bound for $Q_n(q)$ if $q \to 1$. He proved that

$$Q_n(q) \leqslant (1-q^n)^{n-1} \qquad (4)$$

for $n \geqslant 1$. Stepanov's proof is quite involved and relies on one of his earlier results (V. E. Stepanov [5].) For (4) a simpler proof was given in 1978 by D. E. Knuth and A. Schönhage [3].

In what follows we shall prove a general inequality related to the connectedness of $\Gamma_n(p)$ which contains (4) as a particular case. In 1985 W. Kordecki [4] generalized the aforementioned results of V. E. Stepanov [5] in another way.

2. A Generalization of Stepanov's Inequality

Let us fix a set of s vertices, say $\{1, 2, \ldots, s\}$, in the graph $\Gamma_n(p)$. We say that vertex j can be reached from vertex i if the two vertices are joined by a path. Define $\rho_n(s)$ as the number of vertices of $\Gamma_n(p)$ which either belong to the set $\{1, 2, \ldots, s\}$ or belong to the set $\{s+1, s+2, \ldots, n\}$ and can be reached from some vertex in the set $\{1, 2, \ldots, s\}$. Clearly, $\rho_n(s)$ is a random variable with possible values $s, s+1, \ldots, n$. The probabilities $\mathbf{P}\{\rho_n(s)=k\}$ for $1 \leqslant s \leqslant k \leqslant n$ can be determined by the recurrence formula

$$\mathbf{P}\{\rho_n(s)=k\} = \sum_{j=0}^{k-s} \binom{n-s}{j} p^j q^{n-s-j} \mathbf{P}\{\rho_{n-1}(s+j-1)=k-1\}, \qquad (5)$$

where $\mathbf{P}\{\rho_0(0)=1\}=1$. To prove (5), let us take into consideration that the event $\rho_n(s)=k$ can occur in several mutually exclusive ways: Vertex 1 is joined by j vertices $(0 \leqslant j \leqslant k-s)$ in the set of vertices $\{s+1, \ldots, n\}$. This event has probability $\binom{n-s}{j} p^j q^{n-s-j}$. Now let us divide the set of vertices $\{2, 3, \ldots, n\}$ into two subsets. The first subset contains $s+j-1$ vertices, namely, the vertices $2, 3, \ldots, s$ and the aforementioned j vertices in $\{s+1, \ldots, n\}$. The second subset contains the remaining $n-s-j$ vertices in $\{s+1, \ldots, n\}$. Then the second subset contains $k-s-j$ vertices which can be reached from vertices in the first subset. This latter event has probability $\mathbf{P}\{\rho_{n-1}(s+j-1)=k-1\}$. The probability that both events occur is equal to the product of the aforementioned two probabilities. If we add these products for every $j=0, 1, \dots, k-s$, then we obtain the desired probability (5).

We observe that

$$\mathbf{P}\{\rho_n(s)=k\}=\binom{n-s}{k-s} p^{k-s} q^{k(n-k)} \Phi_k^{(s)}(q) \tag{6}$$

for $1 \leqslant s \leqslant k \leqslant n$ where $\Phi_k^{(s)}(q)$ is a polynomial of degree $(k+s-3)(k-s)/2$ in q. If we put (6) into (5), we obtain the following recurrence formula for the determination of $\Phi_k^{(s)}(q)$ for $1 \leqslant s \leqslant k$:

$$\Phi_k^{(s)}(q)=\sum_{j=0}^{k-s}\binom{k-s}{j} q^{k-s-j} \Phi_{k-1}^{(s+j-1)}(q). \tag{7}$$

Since $\Phi_0^{(0)}(q)=1$ and $\Phi_k^{(0)}(q)=0$ for $k \geqslant 1$, the polynomials $\Phi_k^{(s)}(q)$, $1 \leqslant s \leqslant k \leqslant n$, are uniquely determined by (7).

THEOREM. *If* $1 \leqslant s \leqslant k \leqslant n$, *we have*

$$\mathbf{P}\{\rho_n(s)=k\} \leqslant \binom{n-s}{k-s} q^{k(n-k)}(1-q^k)^{k-s}. \tag{8}$$

Proof. We shall prove that

$$\Phi_m^{(i)}(q) \leqslant (1+q+\cdots+q^{m-1})^{m-i} \tag{9}$$

for $1 \leqslant i \leqslant m$. Then (8) will follow from (6) and (9). We shall prove by mathematical induction that (9) is true for all $1 \leqslant i \leqslant m$. If $m=1$, then $\Phi_1^{(1)}(q)=1$ and (9) is true. Suppose that (9) is true for $\Phi_v^{(i)}(q)$ where $1 \leqslant i \leqslant v \leqslant m-1$ and $m \geqslant 2$. Then by (7)

$$\begin{aligned}\Phi_m^{(i)}(q) &\leqslant \sum_{j=0}^{m-i}\binom{m-i}{j} q^{m-i-j}(1+q+\cdots+q^{m-2})^{m-i-j}\\ &=(1+q+\cdots+q^{m-1})^{m-i}.\end{aligned} \tag{10}$$

Accordingly, (9) is true for $1 \leqslant i \leqslant m$ where m is any positive integer.

We note that in a similar way we can prove that

$$\Phi_m^{(i)}(q) \leqslant (1+q+\cdots+q^{m-2})^{m-i} \tag{11}$$

for $1 \leqslant i \leqslant m$ and $m \geqslant 2$.

Obviously, the random graph $\Gamma_n(p)$ is connected if and only if $\rho_n(1)=n$.

Thus by (8)

$$Q_n(q) = \mathbf{P}\{\rho_n(1) = n\} \leqslant (1 - q^n)^{n-1} \tag{12}$$

for $n \geqslant 1$. This is Stepanov's inequality. If $n \geqslant 2$, then by (11) we have also

$$Q_n(q) \leqslant (1 - q^{n-1})^{n-1}. \tag{13}$$

3. The Probabilities $\mathbf{P}\{\rho_s(k) = n\}$

The probabilities $\mathbf{P}\{\rho_s(k) = n\}$, $1 \leqslant s \leqslant k \leqslant n$, naturally appear in the investigation of the structures of polymers in chemistry. J. W. Kennedy [2] has derived a recurrence formula for the determination of $\mathbf{P}\{\rho_n(1) = k\}$ and calculated the probabilities $\mathbf{P}\{\rho_n(1) = k\}$ for $k \leqslant 5$. For $1 \leqslant s \leqslant k \leqslant n$ the probabilities $\mathbf{P}\{\rho_n(s) = k\}$ are given by (6) where

$$\Phi_k^{(s)}(q) = \sum_{\substack{j_1 + \cdots + j_k = k - s \\ j_1 + \cdots + j_r < r \ (1 \leqslant r \leqslant k - s)}} \frac{(k-s)!}{j_1!\, j_2! \cdots j_k!} q^{k(k-s) - j_1 - 2j_2 - \cdots - kj_k} \tag{14}$$

for $1 \leqslant s \leqslant k$.

To prove (14) let us take into consideration that in (14), $j_k = j$ where $0 \leqslant j \leqslant k - s$. Then by (14) we obtain that

$$\Phi_k^{(s)}(q) = \sum_{j=0}^{k-s} \binom{k-s}{j} q^{k-s-j} \Phi_{k-1}^{(s+j-1)}(q) \tag{15}$$

for $1 \leqslant s \leqslant k$ where $\Phi_0^{(0)}(q) = 1$ and $\Phi_k^{(0)}(q) = 0$ for $k \geqslant 1$. This is the same recurrence formula as (7) and the initial conditions are also the same. Consequently, (14) holds for all $1 \leqslant s \leqslant k$.

If $q = 1$ in (15), we obtain

$$\Phi_k^{(s)}(1) = \sum_{j=0}^{k-s} \binom{k-s}{j} \Phi_{k-1}^{(s+j-1)}(1) \tag{16}$$

for $1 \leqslant s \leqslant k$ where $\Phi_0^{(0)}(1) = 1$ and $\Phi_k^{(0)}(1) = 0$ for $k \geqslant 1$. Hence it follows by mathematical induction that

$$\Phi_k^{(s)}(1) = sk^{k-s-1} \tag{17}$$

for $1 \leqslant s \leqslant k$.

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References

1. E. N. Gilbert, Random graphs, *Ann. Math. Statist.* **30** (1959), 1141–1144.
2. J. W. Kennedy, Random clumps, graphs, and polymer solutions, *in* "Theory and Applications of Graphs" (Y. Alavi and D. R. Lick, Eds.), pp. 314–329, Lecture Notes in Mathematics, Vol. 642, Springer-Verlag, Berlin, 1978.
3. D. E. Knuth and A. Schönhage, The expected linearity of a sample equivalence algorithm, *Theoret. Comput. Sci.* **6** (1978), 281–315.
4. W. Kordecki, Some recurrence formulas for probability of connectedness of random graphs, *Theory Probab. Appl.* **30** (1985), 630–632.
5. V. E. Stepanov, Combinatorial algebra and random graphs, *Theory Probab. Appl.* **14** (1969), 373–399.
6. V. E. Stepanov, On the probability of connectedness of a random graph $\mathscr{G}_m(t)$, *Theory Probab. Appl.* **15** (1970), 55–67.